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Dynamical critical behaviours of the Ising spin chain: Swendsen–Wang and Wolff algorithms

P L Krapivsky

Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA

E-mail: paulk@bu.edu

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Abstract

We study the zero-temperature Ising chain evolving according to the Swendsen–Wang dynamics. We determine analytically the domain length distribution and various 'historical' characteristics, e.g. the density of unreacted domains is shown to scale with the average domain length as $\langle l\rangle^{-\delta}$ with $\delta = 3/2$ (for the q-state Potts model, $\delta = 1 + q^{-1}$). We also compute the domain length distribution for the Ising chain endowed with the zero-temperature Wolff dynamics.

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1. Introduction

Interesting and relatively poorly understood dynamical critical behaviours occur when statistical–mechanical systems are quenched from a disordered phase to their critical points. For the Ising spin systems, two very popular dynamics were introduced long ago by Glauber [1] and Kawasaki [2]. Glauber and Kawasaki algorithms are the simplest dynamical rules based on local moves—single spin flips for the non-conservative Glauber dynamics and spin exchanges for the conservative Kawasaki dynamics. Glauber's and other single spin-flip dynamics, particularly the Metropolis algorithm, have become a powerful tool for understanding the equilibrium behaviour of the statistical–mechanical systems well away from the critical temperature [3, 4]. The simulation becomes very slow at criticality, however, and to overcome this difficulty non-local moves, such as cluster flips, have been suggested. The Swendsen–Wang [5] and Wolff [6] dynamics are two well-known cluster algorithms that are widely used in elucidating the equilibrium critical behaviour in statistical physics and lattice field theory. The dynamical critical behaviour of these algorithms is an active area of research, see [7] and references therein. This work heavily relies on simulations, e.g., the value of the dynamical critical exponent for the two-dimensional Ising model endowed with the Swendsen–Wang

dynamics is known only numerically; analytic studies of the Swendsen–Wang dynamics have been limited so far to the Ising model on the complete graph, that is to the Curie–Weiss or the mean-field model [8–10].

The purpose of this paper is to investigate the dynamical aspects of the cluster algorithms—particularly the Swendsen–Wang algorithm—in the simplest possible setting, that is in one dimension. The critical temperature is usually $T_c = 0$ for one-dimensional systems. The zero-temperature dynamics can be quite peculiar, for example the Ising spin chain subject to the zero-temperature Kawasaki dynamics freezes [11, 12]. Non-conservative dynamics, however, usually bring the system to the ground state, e.g., the zero-temperature Ising–Glauber chain reaches the ground state in a time $\tau \sim \mathcal{L}^2$ (here $\mathcal{L}$ is the system size), that is the dynamical exponent is $z = 2$ for the Glauber algorithm [1, 13]. We will see that $z = 0$ for the Swendsen–Wang algorithm in one dimension; more precisely, $\tau \sim \ln \mathcal{L}$. The one-dimensional version of the Swendsen–Wang algorithm is also an appealingly simple model that is reminiscent of other important models of phase-ordering dynamics like the time-dependent Ginzburg–Landau equation with no thermal noise (i.e. $T = 0$) [14, 15]. Furthermore, the one-dimensional Swendsen–Wang algorithm provides a useful laboratory to probe not merely the dynamical critical exponent but various much more subtle dynamical characteristics.

This paper is organized as follows. In section 2 we show that the Ising chain endowed with zero-temperature Swendsen–Wang dynamics exhibits scaling with the average length growing exponentially with time. Section 2 also contains the derivation of the domain length distribution and a number of subtle statistical properties of the domains like the density of domains which never flipped. In section 3 we investigate the Ising chain endowed with zero-temperature Wolff dynamics. This model has been previously studied by Derrida and Hakim [16]; here we further analyse the model, and particularly determine the domain length distribution. A summary is given in section 4.

2. Swendsen–Wang dynamics

In one dimension, the Ising spin chain can be thought of as an array of contiguous alternating domains of up and down spins. At zero temperature, the Swendsen–Wang dynamics accounts for randomly choosing a domain and flipping it. The flipping of a domain I implies that it merges with its left and right neighbouring domains I_- and I_+ thus forming a single domain $I_- \cup I \cup I_+$.

2.1. Domain length distribution

Let $N_l(t)$ be the number of domains of length l and $N(t) = \sum_{l \geqslant 1} N_l(t)$ is the total number of domains. The average number of domains that flip in an infinitesimal time interval Δt is equal to $N(t)\Delta t$. In every flipping event three domains merge into a single one, so $N(t)$ changes according to

$$N(t + \Delta t) = N(t) - 2\Delta t N(t). \tag{1}$$

Similarly, $N_l(t)$ evolves according to

$$N_l(t + \Delta t) = N_l(t) - 3\Delta t N_l(t) + \Delta t \sum_{i+j+k=l} \frac{N_i(t)}{N(t)} N_j(t) \frac{N_k(t)}{N(t)}. \tag{2}$$

The term $3\Delta t N_l(t)$ accounts for the loss that occurs when the domain or either of its neighbours is flipped, while the last term on the right-hand side of equation (2) accounts for the gain due to the flipping of a domain of length j followed by immediate merging with two adjacent

domains of lengths i and k. Equation (1) is obviously exact. The linear loss term in (2) is also exact, while the nonlinear gain term is exact only if the sizes of adjacent domains are uncorrelated. However, whenever a domain merges with two adjacent domains, the resulting domain does not acquire any correlation with the neighbours, i.e. correlations are not dynamically generated. (See [15] for a detailed justification in the context of a somewhat similar model, viz the noiseless time-dependent Ginzburg–Landau equation in which domains also merge with their neighbours.) Therefore, if initially the domain lengths were uncorrelated, they remain uncorrelated at all later times and equation (2) is exact.

The total length of the system is $\mathcal{L} = \sum_{l \geqslant 1} l N_l(t)$. In the thermodynamic limit $\mathcal{L} \to \infty$, it is convenient to use the domain length densities $n_l(t) = N_l(t)/\mathcal{L}$ and the (total) domain density $n(t) = \sum_{l \geqslant 1} n_l(t) = N(t)/\mathcal{L}$. From (1) we find that the domain density evolves according to

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -2n. \tag{3}$$

Solving (3) gives $n(t) = n(0)\mathrm{e}^{-2t}$. Therefore the average domain size $\langle l \rangle = 1/n$ increases exponentially. In contrast, the average size exhibits an algebraic growth $\langle l \rangle \sim t^{1/z}$ [17] in most models describing domain coarsening following a quench to zero temperature.

Likewise, the equation for $N_l(t)$ leads to

$$\frac{\mathrm{d}n_l}{\mathrm{d}t} = -3n_l + \sum_{i+j+k=l} \frac{n_i n_j n_k}{n^2}. \tag{4}$$

The form of equation (4) suggests to consider normalized densities $\rho_l(t) = n_l(t)/n(t)$. These quantities satisfy

$$\frac{\mathrm{d}\rho_l}{\mathrm{d}t} = -\rho_l + \sum_{i+j+k=l} \rho_i \rho_j \rho_k. \tag{5}$$

Equations (4)–(5) are mathematically similar to equations describing the three-particle coalescence process [18] and can be solved accordingly. Introducing the generating function

$$R(t, x) = \sum_{l \geqslant 1} x^l \rho_l(t) \tag{6}$$

we convert an infinite system of equations (5) into a single equation

$$\frac{\partial R}{\partial t} = -R + R^3. \tag{7}$$

Solving (7) gives

$$R(t, x) = \frac{\mathrm{e}^{-t} R_0(x)}{\sqrt{1 - (1 - \mathrm{e}^{-2t}) R_0^2(x)}} \tag{8}$$

with $R_0(x) \equiv R(0, x)$. For instance, consider the evolution starting from the highest energy antiferromagnetic state. In this case, $\rho_l(0) = \delta_{l1}$, i.e. $R_0(x) = x$. Inserting this into (8) and expanding in powers of x we obtain $\rho_{2l}(t) \equiv 0$ and

$$\rho_{2l+1}(t) = \mathrm{e}^{-t} \left(\frac{1 - \mathrm{e}^{-2t}}{4} \right)^l \binom{2l}{l}. \tag{9}$$

The original densities read

$$n_{2l+1}(t) = \mathrm{e}^{-3t} \left(\frac{1 - \mathrm{e}^{-2t}}{4} \right)^l \binom{2l}{l}. \tag{10}$$

In the scaling limit $l, t \to \infty$ with the scaling variable

$$L = l\,\mathrm{e}^{-2t} = \text{finite} \tag{11}$$

the densities become

$$n_l(t) = \mathrm{e}^{-4t}\mathcal{F}(L) \qquad \mathcal{F}(L) = \frac{1}{\sqrt{2\pi L}}\exp(-L/2). \tag{12}$$

For other initial conditions it is quite difficult to extract explicit results for ρ_l from the general solution (8) for the generating function. The most natural situation arises when the system at $T = \infty$ is suddenly quenched to $T = 0$. The appropriate initial condition is completely uncorrelated with each spin taking the values ± 1 independently and with equal probabilities. Then $\rho_l(0) = 2^{-l}$, or $R_0(x) = x/(2-x)$ thus leading to a relatively simple exact expression for the generating function. The resulting ρ_l admits a neat expression

$$\rho_l(t) = \tau^l\,{}_2F_1\left(l, \frac{1}{2}; 1; \frac{1-2\tau}{1-\tau}\right) \qquad \tau = \frac{1+\sqrt{1-\mathrm{e}^{-2t}}}{2}. \tag{13}$$

The apparent simplicity of this solution is illusory as (13) involves the hypergeometric function. Fortunately, the most interesting scaling behaviour (12) is universal, i.e. independent of initial conditions (modulo the assumption that the decay $\rho_l(0)$ versus l is sufficiently steep). Therefore in the following we focus on the antiferromagnetic initial condition.

Note that the domain length distribution $n_l(t)$ is beyond the reach of analytical approaches [17] for the majority of models of domain coarsening; e.g., for the Ising chain endowed with zero-temperature Glauber dynamics the distribution $n_l(t)$ is still unknown although a few exact and approximate results are available [19, 20]. The Swendsen–Wang dynamics is obviously more tractable than the Glauber dynamics—we determined $n_l(t)$ for the antiferromagnetic initial condition and an exact scaling expression in the general case.

2.2. Domain number distribution

For the Swendsen–Wang dynamics it is also possible to analytically probe various historical characteristics. The simplest such quantity is the density $m_p(t)$ of domains composed of p 'parent' domains that never flipped. (Each such domain can of course include domains that flipped during the time interval $(0, t)$.) The domain number distribution $m_p(t)$ is formally defined as follows [20]. Initially $m_p(0) = \delta_{p1}$. In every merging event, the central domain flips while the two adjacent domains do not flip; therefore if they have α and β parents, respectively, the resulting domain has $\alpha + \beta$ parents. The domain number distribution $m_p(t)$ evolves as follows:

$$\frac{\mathrm{d}m_p}{\mathrm{d}t} = -3m_p + \sum_{\alpha+\beta=p}\frac{m_\alpha m_\beta}{n}. \tag{14}$$

A solution to these equations has an exponential form

$$m_p = A\,a^{p-1}. \tag{15}$$

This ansatz reduces an infinite system (14) to a couple of differential equations

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -3A \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = n^{-1}A \tag{16}$$

where $n(t) = \mathrm{e}^{-2t}$, see equation (3). Solving (16) subject to $A(0) = 1$, $a(0) = 0$ (implied by the initial condition $m_p(0) = \delta_{p1}$), we finally obtain

$$m_p(t) = \mathrm{e}^{-3t}(1-\mathrm{e}^{-t})^{p-1}. \tag{17}$$

Thus $P = p\,\mathrm{e}^{-t}$ is the scaling variable and the scaling form is simply exponential:

$$m_p(t) = \mathrm{e}^{-3t}\mathcal{G}(P) \qquad \mathcal{G}(P) = \mathrm{e}^{-P}. \tag{18}$$

The survival of a domain with and without merging is characterized by the domain persistence exponents ψ and δ, which were investigated for the Ising–Glauber [20] and Ising–Kawasaki [21] spin chains. (Domain persistence and closely related cluster persistence exponents were also studied for several other models [22–25].) The exponent δ describes the decay of primordial domains

$$m_1 \sim \langle l\rangle^{-\delta}. \tag{19}$$

The exponent ψ counts the average number of parents per domain $\langle p\rangle \equiv \sum pm_p / \sum m_p$:

$$\langle p\rangle \sim \langle l\rangle^{1-\psi}. \tag{20}$$

In the present case, $m_1 = \mathrm{e}^{-3t}$ and $\langle p\rangle = \mathrm{e}^{t}$, while the average domain length is $\langle l\rangle = \mathrm{e}^{2t}$. Therefore

$$\delta = \tfrac{3}{2} \qquad \psi = \tfrac{1}{2}. \tag{21}$$

Even for the simplest models, these exponents are still known only numerically, e.g., $\delta \approx 2.54$ and $\psi \approx 0.252$ for the Ising–Glauber spin chain [20], and $\delta \approx 2.12$ and $\psi \approx 0.39$ for the Ising–Kawasaki spin chain [21]. For a few models, however, the domain persistence exponents have been computed analytically [20, 22, 23], e.g., for the random field Ising–Glauber spin chain $\delta = \infty$ and $\psi = (3-\sqrt{5})/4 = 0.190\,983\ldots$ [23].

2.3. Domain length-number distribution

The (normalized) domain length-number distribution ρ_{lp} captures both the spatial and historical characteristics of the coarsening domain mosaic and contains previous distributions: $\rho_l = \sum_p \rho_{lp}$ and $m_p = n\sum_l \rho_{lp}$. The domain length-number distribution satisfies

$$\frac{\mathrm{d}\rho_{lp}}{\mathrm{d}t} = -\rho_{lp} + \sum_{i+j+k=l}\sum_{\alpha+\beta=p} \rho_{i\alpha}\rho_j\rho_{k\beta}. \tag{22}$$

To determine ρ_{lp} we use the two-variable generating function

$$\mathcal{R}(t,x,y) = \sum_{l\geqslant 1}\sum_{p\geqslant 1} x^l y^p \rho_{lp}(t). \tag{23}$$

Multiplying equation (22) by $x^l y^p$, summing over all l, $p \geqslant 1$ and using already known result (8) for the one-variable generating function $\sum_j x^j \rho_j(t)$, we find that $\mathcal{R}$ satisfies

$$\frac{\partial\mathcal{R}}{\partial t} = -\mathcal{R} + \mathcal{R}^2 \frac{\mathrm{e}^{-t}x}{\sqrt{1-(1-\mathrm{e}^{-2t})x^2}}. \tag{24}$$

Solving this equation subject to $\mathcal{R}(0,x,y) = xy$ we obtain

$$\mathcal{R}(t,x,y) = \frac{\mathrm{e}^{-t}xy}{1-y+y\sqrt{1-(1-\mathrm{e}^{-2t})x^2}}. \tag{25}$$

Expansion in y is simple and for every p we get

$$\sum_{l\geqslant 1} x^l \rho_{lp}(t) = \mathrm{e}^{-t}x\left(1-\sqrt{1-(1-\mathrm{e}^{-2t})x^2}\right)^{p-1}. \tag{26}$$

The expansion in x is also straightforward and it leads to a series representation for ρ_{lp}. Of course, $\rho_{lp} = 0$ for even l's. For odd l's we find

$$\rho_{2l-1,p} = 0 \qquad \text{for} \quad l < p$$

$$\rho_{2p-1,p} = \mathrm{e}^{-t}\left(\frac{1-\mathrm{e}^{-2t}}{2}\right)^{p-1}$$

$$\rho_{2p+1,p} = \frac{1}{2}(p-1)\,\mathrm{e}^{-t}\left(\frac{1-\mathrm{e}^{-2t}}{2}\right)^{p}$$

$$\rho_{2p+3,p} = \frac{1}{8}(p-1)(p+2)\,\mathrm{e}^{-t}\left(\frac{1-\mathrm{e}^{-2t}}{2}\right)^{p+1}$$

etc. To extract the scaling behaviour, it is more convenient to use (25) rather than (26). Writing

$$x = 1-\mathrm{e}^{-2t}\xi \qquad y = 1-\mathrm{e}^{-t}\eta \tag{27}$$

and taking the limit $t \to \infty$ we simplify (25) to

$$\mathcal{R}(\xi,\eta) = \frac{1}{\eta+\sqrt{1+2\xi}}. \tag{28}$$

In the scaling limit $l,\ p, t \to \infty$ with the scaling variables

$$L = l\,\mathrm{e}^{-2t} = \text{finite} \qquad P = p\,\mathrm{e}^{-t} = \text{finite} \tag{29}$$

the domain length-number distribution admits the scaling form

$$\rho_{lp}(t) = \mathrm{e}^{-3t}\mathcal{H}(L,P). \tag{30}$$

Inserting (27), (30) into equation (23) and replacing summation by integration we convert the two-variable generating function into the double Laplace transform

$$\mathcal{R}(\xi,\eta) = \int_0^\infty \mathrm{d}L\,\mathrm{e}^{-\xi L}\int_0^\infty \mathrm{d}P\,\mathrm{e}^{-\eta P}\mathcal{H}(L,P). \tag{31}$$

Using equations (28), (31) and performing the inverse Laplace transform we obtain

$$\mathcal{H}(L,P) = \frac{P}{\sqrt{2\pi L^3}}\exp\left(-\frac{L}{2}-\frac{P^2}{2L}\right). \tag{32}$$

Comparing (32) with individual distributions (12) and (18) we see that the domain length-number distribution does not factorize even in the scaling limit.

From the length-number distribution one can extract a lot of things, e.g. the fraction of persistent spins, i.e. spins which have not flipped during the time interval $(0, t)$; this quantity usually decays as $\langle l\rangle^{-\theta}$, where θ is the persistence exponent [26]. For the antiferromagnetic initial condition, for instance, the number of persistent spins in a domain is exactly equal to the number of parents. The average number of parents is

$$\langle p\rangle_l = \frac{\sum_{p\geqslant 1} p\rho_{lp}}{\sum_{p\geqslant 1}\rho_{lp}} \equiv \rho_l^{-1}\sum_{p\geqslant 1} p\rho_{lp}. \tag{33}$$

In the long time limit, we can use (32) and replace the summation by integration. This leads to the asymptotic

$$\langle p\rangle_l \to \sqrt{\frac{\pi l}{2}} \qquad \text{when} \quad l\to\infty. \tag{34}$$

The fraction of persistent spins $n^{-1}\sum_{l\geqslant 1}\langle p\rangle_l n_l$ is now computed to give $\mathrm{e}^{-t} = \langle l\rangle^{-1/2}$. Thus, the persistence exponent is $\theta = 1/2$.

2.4. The q-state Potts model

Some of the above calculations can be generalized to the case of the Potts model. For the q-state Potts model, each domain is in one of the q possible states, and each time a domain is updated it adopts the state of one of the two adjacent domains. The updating of a domain results in merging of all three domains with probability $1/(q-1)$ while with probability $(q-2)/(q-1)$ only two domains merge. Therefore the average number of domains lost in every merging event is $2 \times \frac{1}{q-1} + \frac{q-2}{q-1} = \frac{q}{q-1}$, leading to

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\frac{q}{q-1}n. \tag{35}$$

Therefore the average domain size $\langle l \rangle = 1/n$ increases exponentially for arbitrary q.

The number distribution $m_p(t)$ is also readily computable for the q-state Potts model. One has

$$\frac{\mathrm{d}m_p}{\mathrm{d}t} = -\frac{q+1}{q-1}m_p + \frac{1}{q-1}\sum_{\alpha+\beta=p}\frac{m_\alpha m_\beta}{n}. \tag{36}$$

This equation admits the exponential ansatz (15) that reduces (36) to a couple of differential equations

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{q+1}{q-1}A \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{A}{(q-1)n} \tag{37}$$

where $n = \exp\left[-\frac{q}{q-1}t\right]$. Solving (37) subject to the initial conditions $A(0) = 1$ and $a(0) = 0$ (implied by $m_p(0) = \delta_{p1}$) we obtain

$$m_p(t) = Q^{q+1}(1-Q)^{p-1} \qquad Q(t) \equiv \mathrm{e}^{-t/(q-1)}. \tag{38}$$

Re-expressing the quantities $m_1 = Q^{q+1}$ and $\langle p \rangle = Q^{-1}$ through the average domain length $\langle l \rangle = n^{-1} = Q^{-q}$ we find that the domain persistence exponents defined via equations (19)–(20) are given by

$$\delta = \frac{q+1}{q} \qquad \psi = \frac{q-1}{q}. \tag{39}$$

The exponents thus obey the sum rule $\delta(q) + \psi(q) = 2$.

Now consider the length distribution. Particularly, the normalized domain length distribution evolves according to

$$\frac{\mathrm{d}\rho_l}{\mathrm{d}t} = -\rho_l + \frac{1}{q-1}\sum_{i+j+k=l}\rho_i\rho_j\rho_k + \frac{q-2}{q-1}\sum_{i+j=l}\rho_i\rho_j. \tag{40}$$

From this equation, we find an implicit relation for the generating function (6),

$$\frac{(1-R)^{q-1}(R+q-1)}{R^q} = \mathrm{e}^{qt}\frac{(1-R_0)^{q-1}(R_0+q-1)}{R_0^q}.$$

This relation is a polynomial of R of degree q. Hence it is impossible to determine an explicit relation for the generating function, $R(t,x) = \mathcal{F}[t, R_0(x)]$, apart from the Ising case ($q = 2$) and next two cases $q = 3, 4$. The explicit expressions for $q = 3, 4$ are very involved so the exact computation of $n_l(t)$ looks daunting.

Rather than seeking an exact solution, let us consider the asymptotic behaviour. The scaling ansatz

$$\rho_l(t) = n\mathcal{F}(L) \qquad L = nl \tag{41}$$

recasts (40) as

$$\mathcal{F} + qL\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}L} + \mathcal{F} * \mathcal{F} * \mathcal{F} + (q-2)\mathcal{F} * \mathcal{F} = 0 \tag{42}$$

where the symbol $*$ denotes the convolution operation, so that $\mathcal{F} * \mathcal{F}$ is the usual convolution $\int_0^L \mathrm{d}L_1 \mathcal{F}(L_1)\mathcal{F}(L-L_1)$, and $\mathcal{F} * \mathcal{F} * \mathcal{F} = \int_0^L \int_0^L \mathrm{d}L_1\, \mathrm{d}L_2 \mathcal{F}(L_1)\mathcal{F}(L_2)\mathcal{F}(L-L_1-L_2)$. The Laplace transform $\Phi(s) = \int_0^\infty \mathrm{d}L\, \mathrm{e}^{-sL}\mathcal{F}(L)$ satisfies

$$qs\frac{\mathrm{d}\Phi}{\mathrm{d}s} = \Phi^3 + (q-2)\Phi^2 - (q-1)\Phi \tag{43}$$

whose (implicit) solution reads

$$s = (1-\Phi)\left[\frac{\Phi + q - 1}{q\Phi^q}\right]^{\frac{1}{q-1}}. \tag{44}$$

The sum rules $\sum \rho_l = 1$ and $\sum l\rho_l = n^{-1}$ lead to

$$\int_0^\infty \mathrm{d}L\, \mathcal{F}(L) = 1 \qquad \int_0^\infty \mathrm{d}L\, L\mathcal{F}(L) = 1. \tag{45}$$

These two constraints determine the first two constants in the small s expansion of the Laplace transform: $\Phi(s) = 1 - s + \cdots$. This behaviour was taken into account in fixing a constant in the general solution to equation (43).

To complete the task, we must find $\Phi(s)$ from (44) and then perform the inverse Laplace transform. The first step therefore requires finding a root of the polynomial of Φ of degree q. Thus it is apparently impossible to find an explicit scaling solution when $q \geqslant 5$. However, we can deduce the most interesting asymptotics for an arbitrary q. For instance, equation (44) yields $\Phi \to (1-q^{-1})^{1/q} s^{-1+1/q}$ as $s \to \infty$, from which

$$\mathcal{F}(L) \to \frac{\left(1-\frac{1}{q}\right)^{\frac{1}{q}}}{\Gamma\left(1-\frac{1}{q}\right)} L^{-\frac{1}{q}} \qquad \text{as} \quad L \downarrow 0. \tag{46}$$

The large L behaviour of $\mathcal{F}(L)$ is reflected by the type of the (closest to the origin) singularity of its Laplace transform. From equation (44) we find that the singularity is the branch point located at $s_* = -q^{-1/(q-1)}$, namely

$$\Phi \to 2^{-\frac{1}{2}} q^{\frac{q-2}{2(q-1)}} \left[s + q^{-\frac{1}{q-1}}\right]^{-\frac{1}{2}}$$

leading to

$$\mathcal{F}(L) \to q^{\frac{q-2}{2(q-1)}} \frac{1}{\sqrt{2\pi L}} \exp\left[-Lq^{-\frac{1}{q-1}}\right] \tag{47}$$

as $L \to \infty$. Thus the large L behaviour is qualitatively similar for all q, while the small L behaviour is affected by the number of states.

The Swendsen–Wang dynamics of the q-state Potts model is particularly simple in the $q \to \infty$ limit when only two adjacent domains can merge. The domain number distribution is trivial in this case, $m_p(t) = \mathrm{e}^{-t}\delta_{p1}$. The normalized domain length distribution satisfies

$$\frac{\mathrm{d}\rho_l}{\mathrm{d}t} = -\rho_l + \sum_{i+j=l} \rho_i \rho_j. \tag{48}$$

This system of equations resembles (14) and the solution is accordingly found by seeking $\rho_l(t)$ in an exponential form like (15). For initially uncorrelated Potts spins, all domains have initial length 1 when $q = \infty$. Therefore $\rho_l(0) = \delta_{l1}$ and the solution reads

$$\rho_l(t) = \mathrm{e}^{-t}(1-\mathrm{e}^{-t})^{l-1}. \tag{49}$$

The scaling form of the domain length distribution is exponential, $\mathcal{F}(L) = \mathrm{e}^{-L}$; of course, this result can also be derived from (44) which in the $q \to \infty$ limit simplifies to $\Phi = (1+s)^{-1}$. An apparent contradiction of the pure exponential scaled domain length distribution and the general large L asymptotic (47) is the indication that the limits $q \to \infty$ and $L \to \infty$ do not commute.

3. Wolff dynamics

At zero temperature, the Wolff dynamics accounts for randomly choosing a spin and flipping the whole domain containing that spin. The flipping of a domain I again implies that it merges with its left and right neighbouring domains I_- and I_+ to form a domain $I_- \cup I \cup I_+$. In contrast with the Swendsen–Wang dynamics, the flipping of a domain now occurs with a rate proportional to its length. Therefore the domain density decreases with constant rate

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -2 \tag{50}$$

i.e. $n(t) = 1 - 2t$ and the whole system reduces to a single domain at $t_c = 1/2$.

The governing equations for the domain length densities read [16]

$$\frac{\mathrm{d}n_l}{\mathrm{d}t} = -ln_l - 2\frac{n_l}{n} + \sum_{i+j+k=l} \frac{jn_i n_j n_k}{n^2}. \tag{51}$$

We again use the normalized densities $\rho_l(t)$. They satisfy

$$\frac{\mathrm{d}\rho_l}{\mathrm{d}t} = -l\rho_l + \sum_{i+j+k=l} j\rho_i \rho_j \rho_k. \tag{52}$$

The generating function (6) satisfies

$$\frac{\partial R}{\partial t} = -x\frac{\partial R}{\partial x} + xR^2\frac{\partial R}{\partial x}. \tag{53}$$

Changing variables from (t, x) to $(T, X) \equiv (t, t - \ln x)$ removes the first term on the right-hand side of equation (53):

$$\frac{\partial R}{\partial T} = -R^2\frac{\partial R}{\partial X}. \tag{54}$$

Rewriting (54) for $X = T(R, X)$ gives

$$\frac{\partial X}{\partial T} = R^2 \tag{55}$$

which is solved to yield $X = F(R) + R^2T$, or

$$t - \ln x = F(R) + R^2 t \tag{56}$$

with $F(R)$ depending on initial conditions. For the antiferromagnetic initial condition $R_0(x) = x$ and thence $F(R) = -\ln R$, so the exact solution (56) becomes

$$x = R\,\mathrm{e}^{t-R^2t}. \tag{57}$$

Clearly, equation (57) gives an expansion of x in terms of R. We are seeking the opposite, $R = \sum \rho_l x^l$. To determine ρ_l we write

$$\rho_l = \frac{1}{2\pi\sqrt{-1}}\oint \mathrm{d}x\,\frac{R(x)}{x^{l+1}} = \frac{1}{2\pi\sqrt{-1}}\oint \mathrm{d}R\,\frac{Rx'(R)}{[x(R)]^{l+1}}$$

which is combined with equation (57) to give

$$\rho_l(t) = \frac{\mathrm{e}^{-lt}}{2\pi\sqrt{-1}} \oint \mathrm{d}R \left[\frac{\mathrm{e}^{lR^2t}}{R^l} - 2t \frac{\mathrm{e}^{lR^2t}}{R^{l-2}} \right]. \tag{58}$$

An elementary computation shows that $\rho_{2l} \equiv 0$ and

$$\rho_{2l+1}(t) = \frac{(2l+1)^{l-1}}{l!} t^l \exp[-(2l+1)t]. \tag{59}$$

Since $n_l = n\rho_l$ and $n = 1-2t$, the densities vanish at $t_c = 1/2$. Note also that in the proximity of the critical point (i.e. when $n \to 0$ and $l \to \infty$), equation (59) simplifies to

$$\rho_l(t) \simeq \pi^{-1/2} l^{-3/2} \exp(-ln^2/4). \tag{60}$$

This expression was previously derived in [16] by a direct analysis of the generating function near the critical point. Equation (60) shows that $\langle l \rangle = n^{-1}$ does not characterize the domain length distribution: almost all domains are short with lengths of order 1 but because the domain length distribution is algebraic, $\rho_l \propto l^{-3/2}$ with a cutoff length of the order of n^{-2}, and the average domain length diverges as n^{-1}.

The single domain covers the entire spin chain at $t_c = 1/2$, i.e. the system undergoes a gelation transition. This transition differs from the ordinary gelation transition that occurs in mean-field models of polymerization [27, 28] and evolving random graphs [29] where the giant component that is born at the critical time undergoes a long period of growth before it engulfs the entire system. The difference from the ordinary gelation transition is not surprising: in one dimension, the giant component must cover the entire system, so the transition is discontinuous, while in the mean-field models the transition is continuous. Despite this important physical distinction, the domain length distribution (60) is very similar to the cluster size distribution in polymerization [28] and the component size distribution in evolving random graphs [29].

4. Summary

We demonstrated that the Ising chain endowed with the zero-temperature Swendsen–Wang dynamics exhibits scaling with the average length growing exponentially with time. We computed the domain length distribution in special cases, e.g. for the antiferromagnetic initial condition and the random initial condition. The scaled domain length distribution was shown to be a product of a power law and an exponential over the entire range of the (scaled) length. We also computed the domain number distribution, the domain length-number distribution and the domain persistence exponents. The domain length-number distribution does *not* factorize into the product of the single variable distributions. Some of the calculations have been generalized to the Potts model; in particular, the domain number distribution and the domain persistence exponents have been obtained.

We also studied the Ising chain endowed with the zero-temperature Wolff dynamics and showed that the system undergoes a kind of gelation—below the critical time the total number of domains is an extensive variable (proportional to the system size) while at the critical time the entire system gets covered by the single domain. This gelation transition is discontinuous since in one dimension, the giant component must cover the entire system; in contrast, in the mean-field models the gelation transition is continuous. Still we found that the domain length distribution in the zero-temperature Ising–Wolff chain is mathematically similar to the cluster size distribution in polymerization.

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